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# The small-x Behavior of the Nonsinglet Polarized Structure Function $g_2$ in the Double Logarithmic Approximation

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## Abstract

The nonsinglet spin dependent structure function  $g_2(x, Q^2)$  is studied at small  $x$  within the double logarithmic approximation of perturbative QCD. Both positive and negative signature contributions are considered, and we find a power-like growth in  $1/x$ . We discuss how our result fits into the Wandzura-Wilczek relation.

## 1 Introduction

The  $Q^2$  evolution of the spin dependent structure function  $g_1$  is well known. In order to follow the evolution at  $x \sim O(1)$  one can use the original Altarelli-Parisi equation[1] or the anomalous dimensions calculated in[2, 3]. In the region of very small  $x$  it is possible to use the double logarithmic approximation [4, 5] to examine the small- $x$  behaviour of  $g_1$ . In contrast to this, the situation with other polarized structure functions, in particular  $g_2$ , is not so clear at the moment.

The problem comes from the fact that  $g_2$  has no parton interpretation.  $g_2$  describes the scattering on a transversely polarized nucleon, and in terms of the Wilson operator product expansion (OPE) it corresponds to the off-diagonal element of the density matrix  $\langle N(\lambda') | \hat{O} | N(\lambda) \rangle$  with the helicities  $\lambda' \neq \lambda$ .

Let us recapitulate a number of well known facts about the relation between the functions  $g_1$  and  $g_2$  (see, for example, the review[6]):

- i) in the lowest approximation for the deep inelastic scattering (DIS) on a free (on mass shell) quark, the function  $g_2(x, Q^2) = 0$  vanishes identically ( see[6] for a review);
- ii) in the limit  $m^2 \ll Q^2$ , the "massless" quark may have only the longitudinal polarization; so  $g_1(x, Q^2) + g_2(x, Q^2) = 0$  [7];

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iii) the Wandzura-Wilczek relation[8] states, for the contribution of twist-2 operators, that

$$g_1(x, Q^2) + g_2(x, Q^2) = \int_x^1 \frac{dy}{y} g_1(y, Q^2) ; \quad (1)$$

iv) for the spin dependent structure function  $g_2$  the next (twist-3) contribution is not suppressed, even for large  $Q^2$  [9].

The presence of a twist-3 contribution  $g_2^{(3)}(x, Q^2)$  in  $g_2(x, Q^2)$  can be seen in several ways. First of all, from an explicit one loop calculation [10, 11] it is known that  $g_2$  does not satisfy Eq.(1). Next, turning to the small- $x$  region, it is again evident that the relation (1) cannot be valid for the whole functions  $g_1$  and  $g_2$ :  $g_1$  corresponds to an odd-signature amplitude, while  $g_2$  consists of even and odd signature pieces (see for example [7, 12]). Therefore  $g_1$  and  $g_2$  are expected to have quite different  $x$ -behaviour near  $x = 0$ , i.e. they are proportional to different powers of  $1/x$  at small  $x$  (see [4, 7, 13]).

In earlier papers [2, 14] a simple evolution equation for the moments of twist-3 parts of  $g_2$  ( $g_2^{(3)}(x, Q^2)$ ) function has been written, and the anomalous dimensions have been calculated for the scattering off a quark target. Subsequently, it has been realized [15] that the derivation in Ref. [2, 14] was incorrect, because quark operators and quark-gluon operators of the same twist and quantum numbers mix with each other. In [16] a complete basis of operators has been suggested, and the mixing matrix has been calculated in [16]. Similar calculations have later been performed by several other authors [17, 18, 19]. Next in ref. [20] the anomalous dimensions were obtained using another gauge and renormalization scheme. The authors confirm the result of ref. [16], but they disagree with [19].

The most significant departure from the standard leading-twist evolution equations [2, 14] is that the number of operators which contribute to the structure function  $g_2^{(3)}(x, Q^2)$  is not fixed but increases with the moment index  $n$ . Nevertheless, in the two limits:

i) number of colours  $N_c \rightarrow \infty$ , or

ii) the  $(x)$ -moment  $n \rightarrow \infty$

it has been shown that the quark-gluon operators decouple from the quark operator evolution equation [21], and the asymptotic behaviour of  $g_2(x, Q^2)$  in the region  $1 - x \ll 1$  was derived.

In the region of small  $x \ll 1$  another new feature arises. It is known that at small  $x$  the strong ordering of transverse momenta is violated [22, 13, 23, 4]. Instead of  $k_{i,t}^2 \gg k_{i-1,t}^2$  one has  $k_{i,t}^2 \gg \frac{x_i}{x_{i-1}} k_{i-1,t}^2$  (with  $x_i \ll x_{i-1}$ ). As a result, double log contributions of the form  $(\alpha_s \ln^2 1/x)^n$  appear which cannot be summed up in the framework of the conventional  $\log Q^2$  evolution. For  $g_1$  this point was discussed in detail in [4, 5].

In this paper we study the small- $x$  behaviour of the nonsinglet polarized structure function  $g_2$ , in very much the same way as in [4, 5]  $g_1$  has been investigated. We will sum up all the leading pQCD double logarithms (DL) of the form  $\alpha_s^n \ln^k 1/x \ln^m Q^2$  with  $m \leq n$  and  $k + m = 2n - 1$ .

The outline of the paper is as follows. First in Sect.2 we will consider the non-singlet part of  $g_2$  in the first loop leading log approximation (LLA) and show that for the massive quark target (or for the  $\mu$ -meson in the QED case) in the small  $x$  region  $g_2^{(1)} = 0$ . Section 3 contains a detailed discussion of the two loop DL contribution, and in section 4 we derive the DL evolution of  $g_2$  at small  $x$ . In section 5 we analyse the signature content of  $g_2$ , and in section 6 we discuss the connection of our results with the Wandzura-Wilczek relation (1). In the final section we present a brief summary.

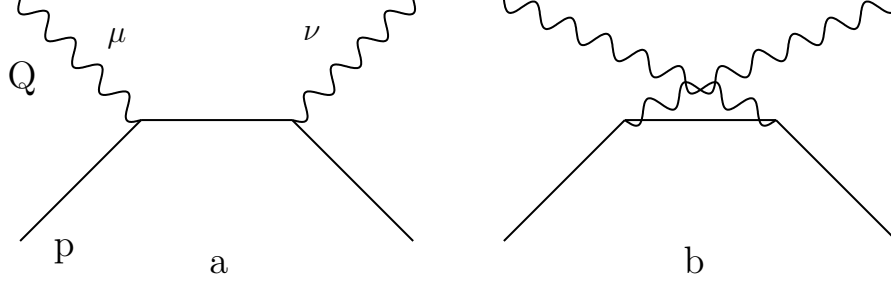


Figure 1: The two Born diagrams

## 2 Definitions and The One Loop Leading Logarithm

We start with a few definitions and general remarks. The spin dependent antisymmetric part of the hadronic tensor  $iW_{\mu\nu}^A$  of the DIS lepton-hadron amplitude has the form:

$$T_{\mu\nu}^A = \frac{M}{(pQ)} i\epsilon^{\mu\nu\alpha\beta} \left[ Q_\alpha s_\beta T_1(x, Q^2) + Q_\alpha \left( s_\beta - \frac{(sQ)}{(pQ)} p_\beta \right) T_2(x, Q^2) \right], \quad (2)$$

where  $M$ ,  $p_\mu$ , and  $s_\mu$  denote the mass, the four momentum and the spin vector of the target, resp. As usual,  $(sp) = 0$  and  $s^2 = -1$ , and the spin-dependent structure functions are defined as the discontinuities of  $T_1$  and  $T_2$ :

$$g_{1,2} = -\frac{1}{2\pi} \text{Im} T_{1,2}. \quad (3)$$

Throughout our paper we work in the Feynman gauge. Note that with the help of the identity [12]

$$\epsilon_{\mu\nu\alpha\beta} p^\alpha s^\beta = \frac{\epsilon_{\mu\nu\alpha\beta}}{2x(pQ)} \left[ (Qs)Q^\alpha p^\beta - (pQ)Q^\alpha s^\beta \right] - (Q_\mu \epsilon_{\nu\alpha\beta\gamma} - Q_\nu \epsilon_{\mu\alpha\beta\gamma}) \frac{p^\alpha Q^\beta s^\gamma}{Q^2} \quad (4)$$

(where the last two terms may be dropped due to the gauge invariance with respect to the photon) in (2) one can replace the vector in front of  $T_2$  by  $-2xp_\alpha s_\beta$ . Then the spin structure corresponding to the function  $g_2$  takes the form  $(2xM)/(pQ)\epsilon^{\mu\nu\alpha\beta} p_\alpha s_\beta g_2$ . It is this form which appears naturally when we calculate the trace of the scattering amplitude.

An important point that we have to discuss is the crossing symmetry of the amplitude. Beginning with the Born approximation we have to sum the two Feynman diagrams Fig.1a and Fig.1b; they will be referred to as the s-channel and the u-channel graphs. The contribution of the u-channel graph may be obtained from the s-channel diagram just by permuting the photon indices  $\mu \rightleftharpoons \nu$  and by replacing  $Q \rightarrow -Q$ . The whole amplitude, given by the sum of Fig.1a and Fig.1b contributions, should be symmetric under this combined  $(\mu \rightleftharpoons \nu, Q \rightarrow -Q)$  transformation (this reflects the Bose statistics of two identical photons in  $t$ -channel). To provide such a property the function  $T_1$  should be *odd*, i.e. it must have negative signature. On the other hand, the crossing symmetry of the function  $T_2$ , which is multiplied by the factor  $(2xM)/(pQ) \cdot \epsilon^{\mu\nu\alpha\beta} p_\alpha s_\beta$  is more complicated and, in fact, cannot simply be read off from (2). To illustrate the difficulty we simply note that, from the first part of the second term in (2) (the piece proportional to  $Q_\alpha s_\beta$ ), we would conclude that  $T_2$  has the same signature as  $T_1$ . If, on the other hand, we make use of the identity (4) and neglect (by gauge invariance) the second term on the right hand side, the spin structure of  $T_2$  in (2) can be replaced by  $-2xp_\alpha s_\beta$ . Now the same argument would lead to the conclusion that

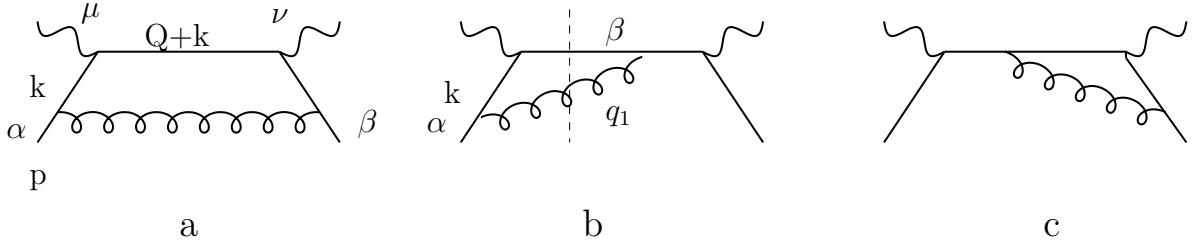


Figure 2: The one-loop diagrams

$T_2$  has to be *even*, i.e.  $T_2$  has positive signature. In section 5 we will show that  $T_2$  has no definite signature but can be written as a sum of two pieces with opposite signature.

To be definite let us consider the high energy scattering of a virtual photon on a heavy fermion with mass  $m$ . In order to obtain  $g_1$  and  $g_2$  we are interested in the energy discontinuity. So in our calculations we will deal with the energy discontinuity, but in order to analyse the signature properties we will have to come back to the full amplitudes  $T_1$  and  $T_2$ .

From the tensor structure of (2) one sees that we have to consider a transversely polarized target. Namely, for a longitudinal polarization vector  $s_\mu = p_\mu$  the vector in front of  $T_2$  vanishes, and only  $T_1$  contributes. For a transverse polarization, on the other hand, both terms in (2) contribute, i.e. the result of such a calculation will be proportional to  $T_1 + T_2$ . We will therefore calculate the energy discontinuity of the scattering amplitude of the transverse polarized target which we will denote by  $g_\perp$ . At the end we then subtract from the result the (already known [4]) function  $g_1$ .

To begin with the Born approximation, we only mention that in the tree approximation the amplitude contributes only to the first term on the r.h.s. of (2). Therefore, the Born value of  $g_2$ ,  $g_{2 \text{ Born}}$ , vanishes. Contributions to the second term will appear only after the  $\alpha_s$  corrections are taken into account.

To study the one loop contributions it is convenient to use Sudakov [24] variables

$$\begin{aligned} k_\mu &= xp'_\mu + \alpha Q'_\mu + k_{t\mu}; \quad d^4k = \frac{s}{2} dx d\alpha d^2k_t; \quad Q'^2 = p'^2 = 0; \\ 2p'Q' &= s; \quad Q_\mu = Q'_\mu - xp'_\mu, \end{aligned} \quad (5)$$

and to separate the transverse and longitudinal parts of the gluon propagator. In the Feynman gauge the spin part of gluon propagator is:

$$d_{\mu\nu} = g_{\mu\nu} = g_{\mu\nu}^\perp + \frac{p'_\mu Q'_\nu + Q'_\mu p'_\nu}{(p'Q')}. \quad (6)$$

For the case of  $g_2$  it is better to deal with the transverse polarization of initial quark, as for the longitudinal quark only the function  $g_1$  gives the leading contribution to the amplitude (2).

So the density matrix of the target quark is:

$$\frac{1}{2}(\hat{p} + m)(1 - \gamma_5 \hat{s}^\perp), \quad (7)$$

and the spin dependent part of the trace in Fig.2a takes the form:

$$Tr^a = -\frac{1}{2} d_{\alpha\beta} Tr \left[ \gamma_\alpha (\hat{p} + m) \gamma_5 \hat{s}^\perp \gamma_\beta (\hat{k} + m) \gamma_\nu (\hat{Q} + \hat{k} + m) \gamma_\mu (\hat{k} + m) \right]. \quad (8)$$

Similarly, for the Fig.2b we obtain

$$Tr^b = -\frac{1}{2}d_{\alpha\beta}Tr \left[ \gamma_\alpha(\hat{p} + m)\gamma_5\hat{s}^\perp\gamma_\nu(\hat{Q} + \hat{p} + m)\gamma_\beta(\hat{Q} + \hat{k} + m)\gamma_\mu(\hat{k} + m) \right], \quad (9)$$

where the quark momentum is  $q_1 = Q + p$ .

It is easy to check that the transverse part of the gluon propagator  $d_{\alpha\beta} = g_{\alpha\beta}^\perp$  gives a vanishing leading log contribution to Eqs.(8) and (9). Indeed, due to the identity

$$\sum_{\alpha\beta} g_{\alpha\beta}^\perp \gamma_\alpha \hat{s}^\perp \gamma_\beta = 0 \quad (10)$$

the expression (8) equals to zero. The situation with Eq.(9) is a bit more complicated.

Note that as we deal with  $s_\mu = s_\mu^\perp$  the polarization vectors of the photons  $(\mu, \nu)$  in (4) have to be different. One has to be longitudinal, whereas the other one is transverse and orthogonal to  $s_\mu^\perp$ . If we chose  $\gamma_\mu = \gamma_\mu^\perp$ ,  $\gamma_\nu = \gamma_\nu^\parallel$  we find, according to (10), that  $Tr^b = 0$ . If, on the other hand,  $\gamma_\nu = \gamma_\nu^\perp$  and  $\gamma_\mu = \gamma_\mu^\parallel$  we arrive at:

$$Tr^b = -Tr \left[ (\hat{p} - m)\gamma_5\hat{s}^\perp\gamma_\nu^\perp(\hat{Q} + \hat{p} - m)(\hat{Q} + \hat{p} - m)(\hat{Q} + \hat{k} + m)\gamma_\mu^\parallel(\hat{k} + m) \right]. \quad (11)$$

At small  $x$ , in the quark propagator we simply put  $1/k^2 \simeq 1/k_t^2$ .

In order to obtain the leading logarithm  $dk_t^2/k_t^2$  we have to keep in (11) only the longitudinal ( $k_\mu \simeq xp$ ) part of quark momentum and, due to the smallness of the quark mass  $m^2 \ll k_t^2$ , one may keep just the first power of  $m$ . Then Eq.(11) takes the form

$$Tr^b \simeq -(1-x)mTr \left[ \hat{p}\gamma_5\hat{s}^\perp\gamma_\nu^\perp\hat{Q}(\hat{Q} + \hat{k})\gamma_\mu^\parallel \right] = 0, \quad (12)$$

as in our LLA limit  $Q + k \simeq Q + xp = Q'$  and  $\hat{p}\hat{Q}\hat{Q}' = -xp^2\hat{Q}' + \hat{p}Q'^2 = -xm^2\hat{Q}' \simeq 0$ ; recall that we neglect the contributions of the order of  $m^2$ .

Now let us consider the longitudinal part of the gluon propagator, i.e. the second term of (6). Keeping only the lowest power of mass the mass  $m$ , and taking into account that

$$d_{\alpha\beta}^\parallel \gamma_\alpha(\hat{p} + m)\gamma_5\hat{s}^\perp\gamma_\beta = 2m\gamma_5\hat{s}^\perp$$

we obtain for the diagram shown in Fig.2a:

$$\begin{aligned} Tr^a &= -mTr \left[ \gamma_5\hat{s}^\perp\hat{k}\gamma_\nu(\hat{Q} + \hat{k})\gamma_\mu\hat{k} \right] = -mk^2Tr \left[ \gamma_5\hat{s}^\perp\gamma_\nu(\hat{Q} + \hat{k})\gamma_\mu \right] \\ &\quad - m2(k s^\perp)Tr \left[ \gamma_5\gamma_\nu(\hat{Q} + \hat{k})\gamma_\mu\hat{k} \right]. \end{aligned} \quad (13)$$

Here one has two quark propagators  $(1/k^2)^2 \approx 1/k_t^4$ . So in order to obtain the leading logarithm we have to keep in the numerator the terms proportional to  $k_t^2$ . After the averaging over the azimuthal angle of the vector  $\vec{k}_t$  the last term in (13) may be written as

$$2m(k s^\perp)Tr \left[ \gamma_5\gamma_\nu(\hat{Q} + \hat{k})\gamma_\mu\hat{k} \right] = mk^2Tr \left[ \gamma_5\gamma_\nu(\hat{Q} + \hat{k})\gamma_\mu\hat{s}^\perp \right] + mk^2Tr \left[ \gamma_5\gamma_\nu\hat{s}^\perp\gamma_\mu\hat{k} \right].$$

Now the first term cancels against the corresponding term in (13). In the last term, within the LLA, we may put  $\hat{k} = x\hat{p}$ , and finally obtain

$$Tr^a = +mxk^2Tr \left[ \gamma_5\hat{s}^\perp\gamma_\nu\gamma_\mu\hat{p} \right]. \quad (14)$$

For the case of Fig.2b the first longitudinal term  $d_{\alpha\beta} = p'_\alpha Q'_\beta / (p'Q')$  gives zero (within the LLA). Thus we have

$$Tr^b = -\frac{1}{2(p'Q')} Tr \left[ \hat{Q}'(\hat{p} + m) \gamma_5 \hat{s}^\perp (\hat{Q} + \hat{p} + m) \hat{p}'(\hat{Q} + \hat{k} + m) \gamma_\mu (\hat{k} + m) \right]. \quad (15)$$

As before, in order to obtain the leading logarithm we put  $\hat{k} = x\hat{p}$ . Next it is useful to note that  $(\hat{Q} + \hat{p} + m) \hat{p}'(\hat{Q} + \hat{k} + m) = 2(p'Q')(\hat{Q}' + m)$ . Therefore

$$\begin{aligned} Tr^b &= -Tr \left[ \hat{Q}'(\hat{p} + m) \gamma_5 \hat{s}^\perp \gamma_\nu \hat{Q}' \gamma_\mu (xp + m) \right] - mx Tr \left[ \hat{Q}' \hat{p} \gamma_5 \hat{s}^\perp \gamma_\nu \gamma_\mu \hat{p} \right] \\ &= -m Tr \left[ \hat{Q}' \hat{p} \gamma_5 \hat{s}^\perp \gamma_\nu \hat{Q}' \gamma_\mu \right] - (2pQ) mx Tr \left[ \gamma_5 \hat{s}^\perp \gamma_\nu \gamma_\mu p \right] - m Tr \left[ Q' \gamma_5 s^\perp \gamma_\nu Q' \gamma_\mu p \right]. \end{aligned} \quad (16)$$

The first term on the r.h.s. of (16) is equal to zero if  $\gamma_\mu = \gamma_\mu^\perp$ . If on the other hand we take  $\gamma_\mu = \gamma_\mu^\parallel$  ( $\gamma_\nu = \gamma_\nu^\perp$ ) we find

$$-Tr \left[ \hat{Q}' \hat{p} \gamma_5 s^\perp \gamma_\nu^\perp Q' \gamma_\mu \right] = 2(pQ) Tr \left[ \gamma_5 s^\perp \gamma_\nu^\perp \gamma_\mu^\parallel Q' \right] = -x(2pQ) Tr \left[ \gamma_5 s^\perp \gamma_\nu^\perp \gamma_\mu^\parallel p \right].$$

To obtain the last equality we have used the fact that the longitudinal photon polarization vector  $e_\mu^\parallel = \frac{1}{\sqrt{Q^2}}(Q' + xp')_\mu$ , i.e.  $\gamma_\mu^\parallel \parallel (Q' + xp')_\mu$ . Therefore in the last expression one can replace  $\hat{Q}'$  by  $-x\hat{p}$ . The factor  $2pQ$  in  $Tr^b$  (15) cancels the rightmost quark propagator  $1/(Q + p)^2 \simeq 1/(2pQ)$ .

Finally, we have to add the contribution of the diagram 2c. After combining it with our results for Figs.2a and b the whole one-loop LLA result takes the form

$$\frac{im}{(pQ)} \epsilon^{\mu\nu\alpha\beta} s_\alpha^\perp Q_\beta \cdot (g_1 + g_2) = \frac{e_q^2 C_F \alpha_s}{8\pi(pQ)} m \int \frac{dk_t^2}{k_t^2} Tr \left[ \gamma_5 \hat{s}^\perp \gamma_\mu \gamma_\nu \hat{Q} \right] [(1 + O(x))], \quad (17)$$

where  $e_q$  is the electric charge of the quark  $q$ ,  $C_F = (N_c^2 - 1)/2N_c$ , and  $\alpha_s$  is the QCD coupling constant.

The well known result for the non-singlet function  $g_1(x, Q^2)$  in the one loop approximation and LLA small  $x$  limit is::

$$g_1 = \frac{e_q^2 C_F \alpha_s}{2\pi} \int \frac{dk_t^2}{k_t^2}. \quad (18)$$

Comparing Eqs.(17) and (18) we see that in this limit  $g_2 = 0 + O(x)$ , in agreement with the more precise computation [10]

$$g_2(x, Q^2) \simeq \frac{e_q^2 C_F \alpha_s}{2\pi} x \ln \frac{Q^2}{m^2 x(1-x)}. \quad (19)$$

### 3 Second loop

Next we turn to the two loop contributions to  $g_2$ . The diagrams are shown in Fig.3. In this section we will show that only the first three diagrams in Figs.3 a -c lead to a double logarithms, whereas in all other diagrams the double logarithm cancels. These cancellations are mainly due to the AFS-mechanism [25, 26]: first one realizes that, in the approximation which leads to the double logarithms, it is enough to keep the nonzero value of the  $x$  variable only in those propagators which have the largest  $\alpha$ . We then find for the integration of either  $x_1$  or  $x_2$  that the propagator poles lie

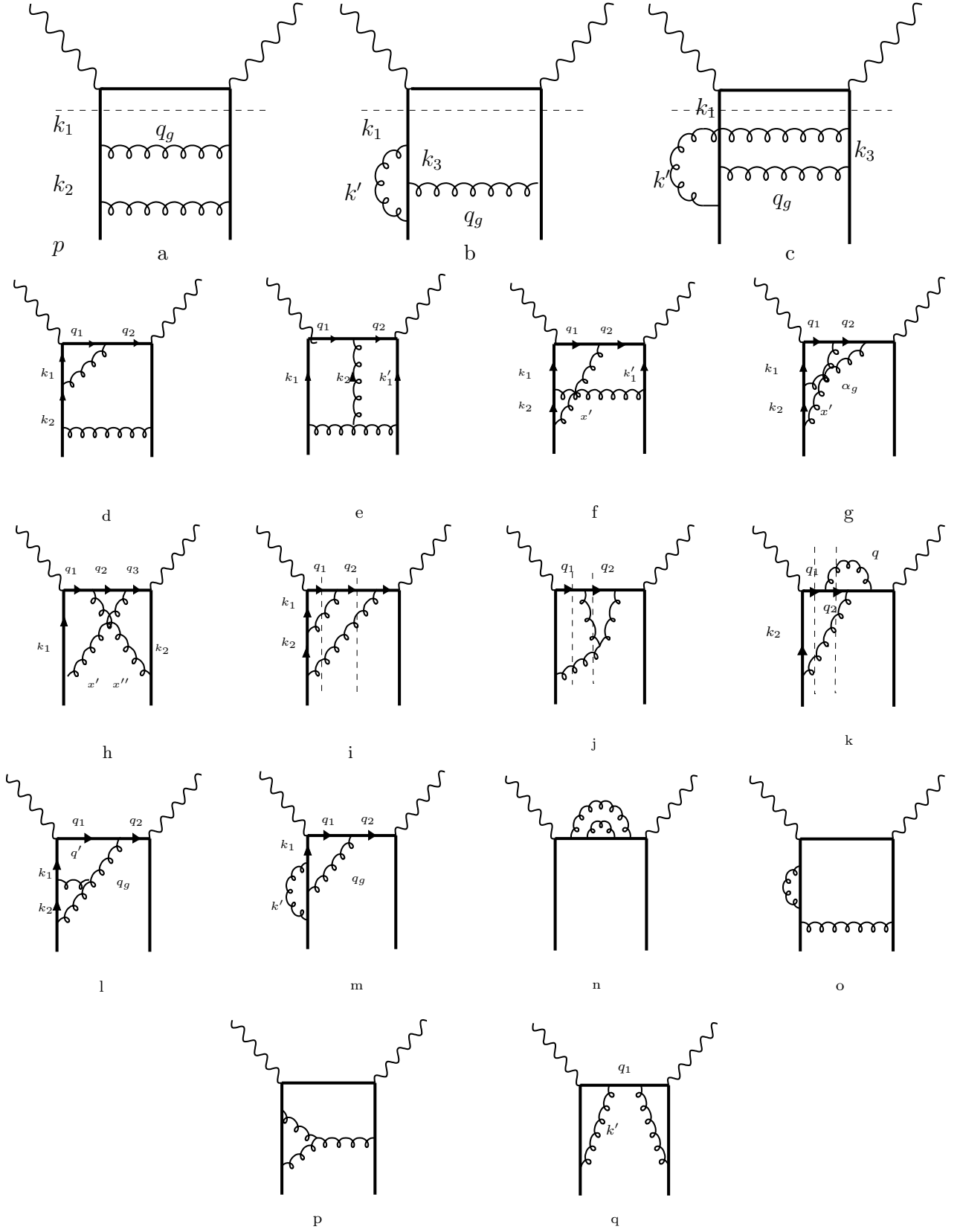


Figure 3: The two-loop diagrams

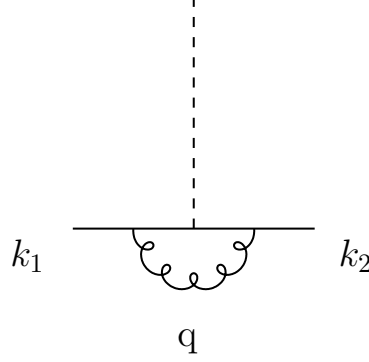


Figure 4: Structure of the Sudakov vertex

on the same side of the integration contour. Their sum adds up to zero, since closing the contour in the opposite half plane (which in this approximation has no propagator poles) gives zero.

To see this in detail, let us go through the diagrams in Fig.3. First we eliminate all self energy insertions to Fig.2a or b: they never change the spin structure and, in the Feynman gauge, the self energy loop has only a single logarithmic divergence. Examples are shown in Figs.3n and 3o.

Next we consider the vertex correction in Fig.3d. The absence of the double logarithm in the last step can be checked by a straightforward computation of the trace (analogous to (9),(15) for the case of Fig.2b). Alternatively, one can use the "AFS" type arguments and consider the cancellation between the cuts passing through the quark lines with  $q_1$  or  $q_2$ .

Next we consider the graphs in Figs.3e-h. In order to obtain the double log contribution we have to consider the domain where all the momenta are strongly ordered. Say, in Fig. 3g  $1 \gg x_2 \gg x_1$ ,  $k_{t2} \ll k_{t1}$  and  $|\alpha_1| \gg |\alpha_2|$ . Let us consider the loop with the largest values of the  $\alpha$  component (the photon momentum fraction). This loop includes 3 quark propagators ( $q_1, q_2$  and  $k_1$ ) and one gluon ( $\alpha_g = \alpha_1 - \alpha_2$ ). Two propagators with the largest  $\alpha_{q_1} \simeq \alpha_{q_2} \simeq 1$  are enough to provide the convergency of the integral over  $dx_1$ . Thus we may neglect all other propagators (with the order of  $\alpha_1$  accuracy). For the  $1/q_1^2$  and  $1/q_2^2$  propagators both poles lie in the same semiplane, and closing the  $x_1$  contour in the opposite semiplane one gets zero result. In other words, the contributions of the  $1/q_1^2$  and  $1/q_2^2$  poles cancel each other. This is analogous to the famous cancellation of the AFS [25] pomeron cut contribution which was first discussed by Mandelstam [26]. This cancellation reflects the fact that the diagram shown in Fig.3g has a wrong space-time structure. In the Breit (photon rest) frame the formation time of the gluon with momentum  $q' = p - k_2$ ,  $\tau' \simeq 1/(\alpha' \sqrt{Q^2})$ , which is proportional to the inverse power of the  $\alpha' \simeq \alpha_2$  component  $\tau' \simeq 1/(\alpha' \sqrt{Q^2})$  is much larger than the formation time of the second gluon with  $q = k_2 - k_1$ , ( $\tau \simeq 1/(\alpha_1 \sqrt{Q^2})$ ), and also much larger than the  $s$ -channel quark time life,  $\tau_q \sim 1/\sqrt{Q^2}$  (the separation between the points  $\mu$  and  $\nu$  in Fig.1). On the other hand, along the upper quark line in Fig.3g we have quite the opposite, wrong time ordering  $\tau_q > \tau > \tau'$ . By the same reason the graphs of Fig.3 e,f graphs give no leading log contribution. In Fig.3e we have  $-\alpha_1 = |\alpha_{k_1}| \gg \alpha'_1$ . Then in the right hand side loop the two propagators with the largest  $\alpha$ -components are  $q_2$  ( $\alpha_q \sim 1$ ) and  $k_2 = q_2 - q_1$  ( $\alpha_2 \sim \alpha_1$ ). In both propagators the photon momentum fraction  $\alpha$  goes in the same direction, i.e. we have both poles in the same  $x_2$ -semiplane, and closing the contour of the  $x_2$  integration in the opposite semiplane one obtains zero. In the case of large  $\alpha'_1 \gg -\alpha_1$  (of course, we still have  $\alpha'_1 \ll 1$  and  $-\alpha_1 \ll 1$ ) we obtain the same zero result, due to the  $x_1$ -integration in the left hand side loop. Finally, for the diagrams Fig.3f and h we consider the upper  $(k_1, q_1, q_2, k'_1)$  loop (without the quark  $k_2$ ): the two

propagators with the largest photon momentum fraction  $\alpha$  are  $q_1$  and  $q_2$ , and we have the same cancellation as for the case of Fig.3g.

The diagrams in Figs.3i-k contain vertex corrections as subgraphs. The general structure of such a ‘Sudakov’ vertex is illustrated in Fig.4. The double logarithmic (DL) contribution has been analysed first in [24]. It comes from the region of  $\alpha_1 \gg \alpha_q \gg \alpha_2$ ,  $x_1 \ll x_q \ll x_2$  (for the case of  $\alpha_1 \gg \alpha_2$ ,  $x_1 \ll x_2$ ). The corresponding kinematical domain for the graphs of Fig.3 i,j,k is  $1 \gg \alpha_q \gg \alpha_2$ , with  $q = k_2 - k_1$  in Fig.3i and  $q = q_1 - q_2$  in Fig.3j,k. Indeed, say, in the upper triangle of Fig.3i one may close the  $x_1$ -contour around the  $1/q^2$  pole of the gluon propagator. Then the quark propagators  $1/k_1^2 \approx 1/(x_2 \alpha_q s)$  (here  $x_1 = x_2 - x_q \approx x_2$  as  $x_q \ll x_2$ ) and  $1/q_1^2 \simeq 1/x_q s$ . To obtain the leading logarithm one has to consider the longitudinal polarization of gluons  $d_{\alpha\beta}^{\parallel} = Q'_\alpha p'_\beta / (Q' p')$  (as we have seen in Sec.2). In the trace corresponding to the quark loop it gives  $\hat{Q}' \hat{k}_2 \hat{Q}' = x_2 s \cdot \hat{Q}'$  and  $\hat{p}' \hat{q}_2 \hat{p}' = s \cdot \hat{p}'$ . In other words, we reproduce the original spin structure of the diagram Fig.2b and obtain the factor  $x_2$  which cancels the  $1/x_2$  coming from the  $1/k_1^2$  propagator. Thus the loop integral

$$\int \frac{dk_{1t}^2 d\alpha_q dx_q}{\alpha_q x_q} \delta(\alpha_q x_q s + (k_1 - k_2)_t^2) = \int \frac{d\alpha_1}{\alpha_q} \frac{dx_q}{x_q} \quad (20)$$

takes the double logarithmic form. However in such a domain for the second loop we have the ‘AFS’-type cancellation again. In Fig.3i for the loop with 2 gluons and quarks  $k_2$  and  $q_2$  the two largest  $\alpha$  ( $\alpha_q$  and  $\alpha_{q_2} \simeq 1$ ) go in the same direction, leading to two poles in the same  $x_2$  semiplane. In the left loop of Fig.3j (which does not include the quark  $q_2$ ) the two largest  $\alpha$  variables are  $\alpha_{q_1}$  and  $\alpha_q$  which flow in the same direction as well. For the case of Fig.3k one has the same cancellation as in Fig.3g. In the lower triangle (with 3 quark and one gluon propagators) the two largest  $\alpha$  are  $\alpha_{q_1} \simeq 1$  and  $\alpha_{q_2} \simeq 1$ . Here (in the vertex correction) the absence of a DL contribution may be treated as a well known cancellation of the leading logs between the virtual (Sudakov formfactor) and real ‘soft’ gluon emission in the inclusive cross section; i.e. the cancellation between the cuts shown by the left- and right-dotted lines in Fig.3 i,j,k.

For the case of the vertices shown in Fig.3l, it is more convenient to fix the value of  $x'$  within the conventional DL region ( $1 \gg x' \gg x_1$ ) and then to observe the ‘AFS’-type cancellation in the upper loop ( $k_1, q_1, q_g$ , but without the  $k'$ -propagator) due to the two largest fraction of the target momentum  $x_{q'} = x' - x_1$  and  $x_g \approx 1(q_g = q_2 - q_1 = x_g p + \alpha_g Q' + q_t)$  which go in the same direction.

The vertex Fig.3m does not give the double log at all. For  $x' \ll 1$  and  $\alpha' \ll \alpha_1$  the virtuality  $(k' - k_1)^2$  of the quark propagator is controlled not by the value of  $\alpha_1 x' s$  but by  $k_{1t}^2 \sim \alpha_1 s$ <sup>3</sup>. Finally, in the case of the two photon vertex corrections (Fig.3q) one has to ‘cut’ one of the vertices in order to have a small  $x$ . Then the central quark virtuality is  $q_1^2 \simeq s$ , and again we lose the double logarithm in the second vertex, since the virtuality  $(q_1 - k')^2 \simeq s$  of the next quark propagator is not controlled by the gluon loop  $(\alpha', x')$  variables. The same is true for the diagrams in Figs.3j and 3r.

Finally we return to the ‘reducible’ graphs of Fig.3a-c. They all have the property that they can be divided into two parts by cutting two t-channel quark lines. First of all we will check the cancellation of the double logarithms corresponding to the lower (below the crosses) part of the graphs in Fig.3b and c, i.e. the cancellation at the level of the quark-quark scattering amplitude.

<sup>3</sup>If instead of the gluon  $q_g$  one cuts the quark lines  $(p - k')$  and  $(k_1 - k')$  (as in Fig.5a) we lose the logarithm in the upper loop (with gluon  $q_g$ ). Indeed, the quarks  $(p - k')$  and  $(k_1 - k')$  are now on mass shell, and thanks to the gauge invariance one may replace the polarization vector  $Q'$  (at the lower end of the gluon  $q_g$ ) by  $-q_{gt}/\alpha_g \simeq -k_{1t}/\alpha_1$ . This factor  $(k_{1t}/\alpha_1)$  kills the normal Dlog in the loop.

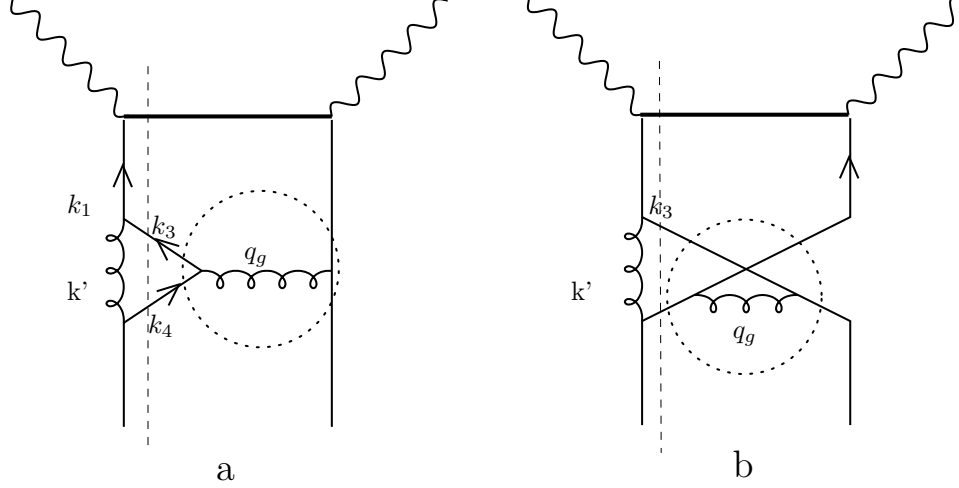


Figure 5: Illustration of the Signature properties of Figs.3b and c

Such a cancellation was first observed in QED [22] and was then discussed in [13] for the DL quark-quark amplitude in QCD.

It is known that the DL contribution comes from the longitudinal  $d_{\alpha\beta}^{\parallel}$  spin part of the gluon  $k'$  propagator in the region of  $x \ll x' \ll 1$  and  $|\alpha'| \ll |\alpha_1|$ . The contour of the  $x'$  integration may be closed around the quark  $1/k_3^2$  pole, and then the contour of the next  $x_g$  integral (see Fig.5a,b) should be closed around the  $1/k_4^2$  pole (or one has to use the  $1/(k_4^2 + i\epsilon)$  propagator to take the imaginary part of the whole amplitude). This gives the "Sudakov"-type double logarithm

$$\int \frac{d\alpha'}{\alpha'} \frac{dk'^2}{k_t'^2} \quad \text{or} \quad \int \frac{d\alpha'}{\alpha'} \frac{dx'}{x'},$$

similar to Eq.(20), but using the  $\delta$ -function  $\delta(k_3^2)$  with  $k_3^2 \approx \alpha_1 x' s + k_{3t}^2$  for the  $d^2 k_t'$  integral ( $\vec{k}_{3t} = \vec{k}'_t - \vec{k}_{1t}$ ) rather than the  $s dx'$  integral. In terms of the  $s$ -channel intermediate states it means that we consider the cut of Fig.3b,c diagrams shown in Fig.5a,b by the dash-dotted line; the upper part of Fig.3b,c is shown in Fig.5 by dashed lines. Note that in Fig.5 we have crossed the upper quark line in order to plot both the graphs in the same form. The even signature (corresponding to  $g_2$  structure function) amplitude is invariant under the crossing transformation, and the only difference between Fig.5a and 5b (or 3b and 3c) comes from the colour charge of the  $k_3$  quark line. The two  $t$ -channel quarks form a singlet colourless state and thus should have an opposite colour sign. Therefore, within the DL accuracy Fig.3b and 3c, diagrams cancel each other.

Note that such cancellation takes place not only in the case of Fig.3b,c but for any DL quark-quark amplitude (in Fig.5 the quark-quark amplitude is denoted by the dotted circle). This cancellation, i.e. the absence of the non-ladder graphs in the DL even signature quark-quark QCD amplitude was proven and discussed in more detail in [13].

As a result of all these cancellations, the only DL two loop contribution to the even signature amplitude comes from the simple diagram in Fig.3a with the two-quark  $t$ -channel intermediate state. However, the calculation of this pure ladder graph Fig.3a is not so trivial, as we have to take care of the longitudinal parts of the quark momenta  $k_1$  and  $k_2$ . For example, even in the DL kinematics the product of 4 vectors  $2(k_1 k_2) = k_1^2 + k_2^2 - q_g^2 = k_1^2 + k_2^2 \neq 2(k_{1t} \cdot k_{2t})$ ; here we put the gluon  $q_g^2 = 0$  on mass shell. That is why we will describe the algorithm of the computation in the next section.

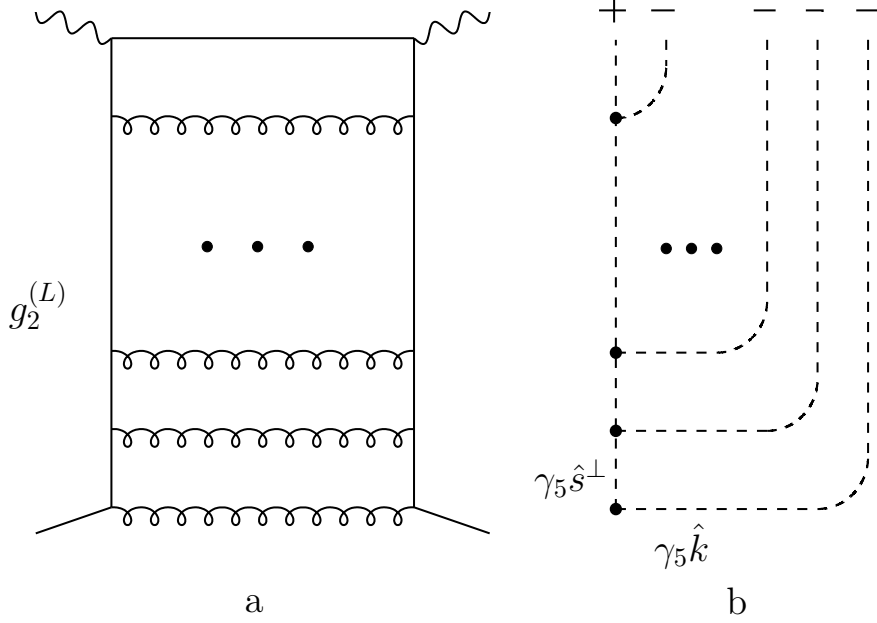


Figure 6: Branches in the ladder graphs

## 4 Evolution of Ladder Graphs

As we have shown, in the two loop approximation only the diagrams Fig.3a-c give double logarithms. In this section we generalize our calculation of the ladder graph (Fig.3a) to all orders (Fig.6). In the next section we will then return to the non-ladder graphs in Fig.3b,c.

We begin at the bottom part of Fig.6a and work our way upwards. After the first gluon rung the spin dependent part of the density matrix (7) takes the form

$$\sum_a \gamma_a \frac{\hat{p} + m}{2} \gamma_5 s^\perp \gamma_a = m \gamma_5 \hat{s}^\perp. \quad (21)$$

As before we keep only the first power of mass  $m \ll \sqrt{Q^2}$ , and therefore one may neglect the mass in all other propagators. Then we write the evolution kernel as

$$\frac{C_F \alpha_s}{2\pi} \int \frac{dx}{x} \frac{dk_t^2}{k^4} \hat{\mathcal{K}}(\gamma_5 \hat{s}) \quad (22)$$

with

$$\hat{\mathcal{K}}(\gamma_5 \hat{s}) = \hat{k} \gamma_5 \hat{s} \hat{k} = (k^2) \gamma_5 \hat{s} - 2(s k) \cdot \gamma_5 \hat{k}. \quad (23)$$

The first term on the r.h.s. of (23) repeats the initial spin structure  $(\gamma_5 \hat{s})$  while the second one has the new matrix structure  $\gamma_5 \hat{k}$ . When upper rungs are included, this second term generates a “new branch” (Fig.6b). Let us first follow this new branch. As there are only two external momenta ( $Q$  and  $k$ ) available for this new term, the subsequent DL evolution with  $n - 1$  rungs gives a contribution of the form

$$2(s^\perp k) \cdot \epsilon^{\mu\nu\alpha\beta} k_\alpha Q_\beta \cdot f^{(n-1)}(k, Q) \quad (24)$$

where the tensor structure comes from the final trace, and the scalar function  $f^{(n-1)}(k, Q)$  is equal (in  $(n-1)$ th order  $\alpha_s$ ) to the ladder part of the structure function  $g_1(x', Q^2)$  (with  $x' = Q^2/2(k \cdot Q)$ ).

Since the first rung in (22) and (23) has the same color factor  $C_F\alpha_s/2\pi$  as the evolution kernel in the ladder part of the longitudinal structure function (see below), we can absorb this rung into  $f^{(n-1)}$  and obtain  $f^{(n)}$ . Finally, from the integration over the azimuthal direction of the vector  $k_t$  the product  $2(s^\perp k)\epsilon^{\mu\nu\alpha\beta}k_\alpha Q_\beta$  turns into  $k_t^2\epsilon^{\mu\nu\alpha\beta}s_\alpha^\perp Q_\beta$ . In this way we obtain the factor  $k_t^2$  which is needed for the logarithmic integral  $k_t^2 dk_t^2/k^4$  in the lowest cell of the ladder. As a result, after summation over  $n$ , the “new branch” resulting from the second term in (23) leads to a contribution of the form [4]:

$$\begin{aligned} g^L(x, Q^2) &= \frac{e_q^2}{2} \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} \left(\frac{1}{x}\right)^\omega \left(\frac{Q^2}{\mu^2}\right)^{f_0(\omega)/2} \frac{\omega}{\omega - f_0(\omega)/2} \\ &= \frac{e_q^2}{2} \delta(1-x) + \frac{e_q^2}{2} \cdot \frac{C_F\alpha_s}{2\pi} \ln \frac{s}{\mu^2} + \sum_{n=2}^{\infty} (C_F\alpha_s)^n g_L^{(n)}(x, Q^2), \end{aligned} \quad (25)$$

where instead of the amplitude  $f_0^{(-)}$  in [4] we now have put the even signature quark-quark amplitude:

$$f_0(\omega) = \omega - \sqrt{\omega^2 - \frac{2C_F}{\pi} \alpha_s}, \quad (26)$$

which describes the pure ladder DL evolution and does not contain the nonladder graphs.

Let us now return to the “normal” branch in (23) (Fig.7) with the structure  $\gamma_5 \hat{s}^\perp$ . At the next-to-lowest rung we simply repeat the argument in (23): we again find, in addition to the ‘old’ structure, the beginning of a new one branch which, after multiplication with the two lower rungs and summation over  $n$ , again leads to a contribution (25). Moving up the ladder in Fig.6, this procedure repeats itself at each rung, i.e. for each order  $\alpha_s$  we obtain a contribution (25). Formally, this multiplicity of (25) is achieved by taking the derivative with respect to  $\alpha_s$  and then multiplying with  $\alpha_s$ :

$$- \alpha_s \frac{\partial}{\partial \alpha_s} g^L \quad (27)$$

where the minus sign reflects the sign of the second term in (23).

Finally we have to complete the ‘normal’ branch which is obtained by following, at each rung, the tensor structure  $\gamma_5 \hat{s}^\perp$ . As we have already said before, for each rung we collect the same coefficient as in the case of the longitudinal target polarization <sup>4</sup>. Having reached the upper end of the ladder, we arrive at the “coefficient function” of the “normal branch”  $m\gamma_5 \hat{s}^\perp$ , which results from taking the trace at the upper end of the ladder. This trace

$$Tr [\gamma_5 \hat{s}^\perp \gamma_\nu (Q + k) \gamma_\mu] \simeq Tr [\gamma_5 \hat{s}^\perp \gamma_\nu (Q + xp) \gamma_\mu]$$

is by a factor 1/2 smaller than the trace  $Tr[\gamma_5 s^\perp \gamma_\nu Q \gamma_\mu]$  in the case of the function  $g_1$ . The DLA result for the (ladder part of) spin dependent non-singlet structure function is given in (25). So the result for the “normal branch” is simply  $\frac{1}{2}g^L$ .

<sup>4</sup>For the longitudinal case the analogue of (23) is  $\hat{k}\gamma_5 \hat{p}\hat{k} = (k^2)\gamma_5 \hat{p} - 2(pk)\gamma_5 \hat{k}$ . In the leading-log kinematics we have  $2(pk) = \alpha_s \approx k_t^2$ , and in the last term we have to keep only the longitudinal part of  $\hat{k} \simeq x\hat{p}$ . Thus one may neglect this last  $(2(pk)\gamma_5 \hat{k} \simeq k_t^2 \cdot x\gamma_5 \hat{p})$  term in comparison with the first  $(k^2 \cdot \gamma_5 \hat{p})$  one, and there is no ‘new branch’ in the longitudinal case.

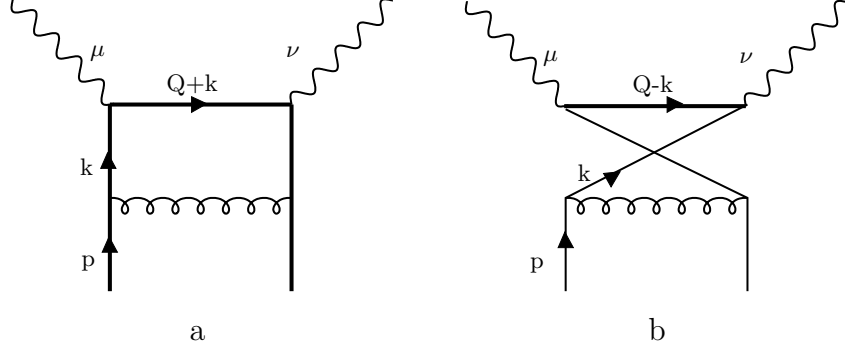


Figure 7: Signature structure of the first loop

Taking the sum of the all contributions, we arrive at our final result for the ladder graphs in Fig.6:

$$g_{\perp}^L = \left( \frac{1}{2} - \frac{\partial}{\partial \ln \alpha_s} \right) g^L(x, Q^2) \quad (28)$$

Since we have studied the transverse polarization of the target, our result represents the ladder-graph contribution to the sum  $g_1 + g_2$ . Before we subtract  $g_1$ , we first turn to the question of signature and the nonladder graphs.

## 5 Signature and the Non-Ladder Diagrams

So far we have generalized our two loop calculation only to the summation of ladder graphs. Now we have to consider the non-ladder graphs Figs.3b and c and discuss their generalization to higher orders. In section 3 we have argued that the double logarithms of Figs.3b and c cancel if we take the even signature combination of Figs.3b, c and their  $s - u$  crossed counterparts, but add up if we consider the odd signature combination (Fig.7). In this section we will analyse the signature content of the ladder graphs: we will find that a general ladder graph  $\mathcal{F}$  of Fig.6 contains both even and odd signature pieces. For the even signature contributions we then know that the nonladder contributions cancel, and only the odd signature pieces of the ladder graphs will receive extra contributions from the nonladder graphs. The summation of the nonplanar diagrams is done in the usual way of [13, 4, 5].

Let us therefore take a closer look at the signature properties of the ladder graphs. For this it is useful to discuss the amplitudes rather than the energy discontinuities. At the bottom of Fig.6 we have the density matrix (21),  $m\gamma_5\hat{s}$ . The contributions of Figs.7a and b to the amplitudes  $T_1$  and  $T_2$  are proportional to:

$$\frac{1}{2Qk + Q^2 + k^2} \text{Tr} \left[ \hat{k}\gamma_5\hat{s}\hat{k}\gamma_{\nu}(\hat{Q} + \hat{k})\gamma_{\mu} \right] \quad (29)$$

and

$$\frac{1}{-2Qk + Q^2 + k^2} \text{Tr} \left[ \hat{k}\gamma_5\hat{s}\hat{k}\gamma_{\mu}(-\hat{Q} + \hat{k})\gamma_{\nu} \right]. \quad (30)$$

As before, we make use of eq.(23)  $\hat{k}\gamma_5\hat{s}\hat{k} = (k^2)\gamma_5\hat{s} - 2(sk)\gamma_5\hat{k}$  (which is valid for any spin vector  $s_{\mu}$ , not only for  $s_{\mu}^{\perp}$ ). Concentrating on the double logarithmic region  $x_k \gg x_{bj} = Q^2/s$ , in the

denominators we retain only the  $2Qk$  term. The contributions (29) and (30) then take the form:

$$\frac{1}{2Qk} \text{Tr} \left[ k^2 (\gamma_5 \hat{s} - 2(sk) \gamma_5 \hat{k}) \gamma_\nu (\hat{Q} + \hat{k}) \gamma_\mu \right] \quad (31)$$

and

$$- \frac{1}{2Qk} \text{Tr} \left[ k^2 (\gamma_5 \hat{s} - 2(sk) \gamma_5 \hat{k}) \gamma_\nu (\hat{Q} - \hat{k}) \gamma_\mu \right] \quad (32)$$

(in the last equation we have written the  $\gamma$ -matrices in the same order as in (31)). One immediately sees that for the first piece ( $\gamma_5 \hat{s}$ ) two contributions with opposite signature emerge: the first one (proportional to  $\text{Tr}[\gamma_5 \hat{s} \gamma_\nu \hat{Q} \gamma_\mu]$ ) has odd signature, whereas the second one (proportional to  $\text{Tr}[\gamma_5 \hat{s} \gamma_\nu \hat{k} \gamma_\mu]$ ) has even signature. The second term ( $\sim 2(sk) \gamma_5 \hat{k}$ ) again leads to two contributions, but the second one vanishes (it has two identical  $\hat{k}$  vectors, and the trace with the  $\gamma_5$  matrix vanishes). So we have only  $2(sk) \text{Tr}[\gamma_5 \hat{k} \gamma_\nu \hat{Q} \gamma_\mu]$ , and this structure has odd signature. Since this is the term which generates the ‘new branch’ we conclude that the ‘new branch’ is purely odd, whereas the ‘old branch’ contains both signatures.

Next we have to relate this decomposition to the tensor structures appearing in (2). Let us first demonstrate explicitly that  $g_1$  has only odd signature (cf. the discussion after (4)). For this we briefly return to the longitudinal polarization of the target. In footnote 3 we have argued that for  $s_\mu = p_\mu$  the second term in the decomposition (23),  $2(pk) \gamma_5 \hat{k}$ , can be neglected in comparison with the first one,  $\gamma_5 \hat{p}$ . In the remaining term  $\text{Tr}[\gamma_5 \hat{p} \gamma_\nu (\hat{Q} + \hat{k}) \gamma_\mu]$  we neglect the  $\hat{k}$  term, since in  $k = x_k p + \alpha_k Q' + k_t$  the first piece gives zero contribution, and the second and the third one destroy the logarithmic divergence of the  $k_t$ -integral. So, indeed, the longitudinal polarization is entirely odd signature, and  $g_1$  has only odd signature.

With this knowledge we return to the transverse target polarization and show how the various pieces resulting from (31) contribute to the tensors in (2). The first term containing  $\hat{Q}$  leads to the spin structure:  $\text{Tr}[\gamma_5 \hat{s} \gamma_\nu \hat{Q} \gamma_\mu] = 4i\epsilon_{\mu\nu\alpha\beta} Q^\alpha s^\beta$  which contributes to both  $g_1$  and  $g_2$ . As said before, it has odd signature. The second (even signature) term has the spin structure of  $T_2$  ( $g_2$ ):  $\text{Tr}[\gamma_5 \hat{s} \gamma_\nu \hat{k} \gamma_\mu] = 4i\epsilon_{\mu\nu\alpha\beta} k^\alpha s^\beta$ , and in the DLA we approximate  $k_\mu \approx x_k p_\mu$  (cf. the discussion after (4)). Finally the spin structure of the last (odd signature) piece,  $2(sk) \text{Tr}[\gamma_5 \hat{k} \gamma_\nu \hat{Q} \gamma_\mu]$ . In order to have two logarithms we need a logarithmic integration  $k_t^2 dk_t^2 / (k_t^2)^2$ . For the transverse polarization  $s_\mu = s_\mu^\perp$  we have already a  $k_t$  factor from the prefactor  $(sk)$ , and in the trace we therefore have to retain the transverse component of  $\hat{k}$ . Recall that this piece contributes to  $g_\perp = g_1 + g_2$ . On the other hand, for the longitudinal polarization  $s_\mu = p_\mu$  we put  $(sk) = (pk) = \alpha_k s/2 = k_t^2/2$ , and inside the trace we substitute  $\hat{k} = x_k \hat{p}$ . This contribution is therefore smaller by the factor  $x_k \approx x_{bj}$  than the ‘normal’ double logarithmic result for the longitudinal polarization. As a result, this last (odd signature) piece is negligible inside  $g_1$  and contributes only to  $g_2$ . Combining all these results we arrive at the conclusion that  $g_1$  has only odd signature contributions whereas  $g_2$  contains both even and odd signature pieces.

To complete our discussion we have to consider diagrams with one more rung (Fig.8). From the density matrix at the bottom we now have not only the ‘old’ matrix  $\gamma_5 \hat{s}$  but also a term of the form  $(sl) \gamma_5 \hat{l}$ . For the former we can repeat all previous steps. For the latter, however, it is nontrivial to show that, at the double log level, the term  $(sl) \text{Tr}[\hat{k} \gamma_5 \hat{l} \hat{k} \gamma_\nu \hat{k} \gamma_\mu]$  does not contribute to the even signature case (otherwise not only  $g_2$ , but also  $g_1$  would receive an even signature contribution). The even signature trace term is written in the form

$$(sl) \text{Tr}[\hat{k} \gamma_5 \hat{l} \hat{k} \gamma_\nu \hat{k} \gamma_\mu] = k^2 (sl) \text{Tr}[\gamma_5 \hat{l} \gamma_\nu \hat{k} \gamma_\mu]. \quad (33)$$

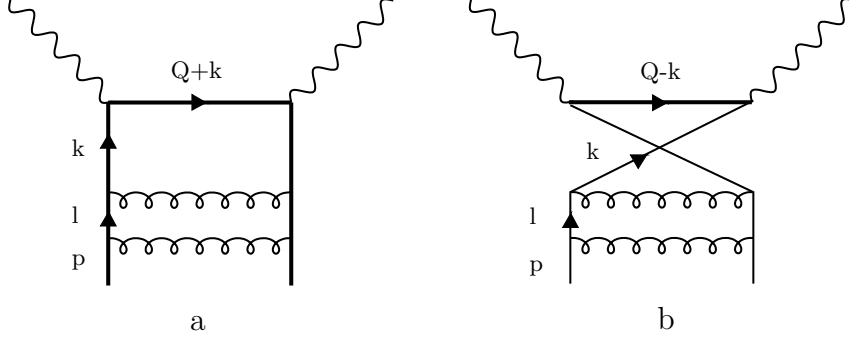


Figure 8: Signature structure of the second loop

One possibility for obtaining the integral  $dl_t^2/l_t^2$  is to keep the transverse components in both  $l$ -factors. This gives:

$$l_t^2 \frac{k^2}{2} \text{Tr}[\gamma_5 \hat{s} \gamma_\nu \hat{k} \gamma_\mu] \approx l_t^2 \frac{k^2}{2} \text{Tr}[\gamma_5 \hat{s} \gamma_\nu x_k \hat{p} \gamma_\mu]. \quad (34)$$

The other possibility is to retain, in the trace, the large ( $x_l \gg x_k$ ) longitudinal component of  $l_\mu$ . Instead of the rhs of (34) we obtain:

$$k^2 (sl) \text{Tr}[\gamma_5 x_l \hat{p} \gamma_\nu \hat{k} \gamma_\mu]. \quad (35)$$

At first sight this expression does not seem to have a double logarithm from the  $l_t$  integral. On the other hand, it is larger than (34) by the ratio  $x_l/x_k$ , and we have to remember that there are corrections from the propagators  $1/k^2$ . Putting  $k^2 = \alpha_k x_k s + k_t^2$  with  $\alpha_k = \frac{(k-l)_t^2}{x_l s}$  we expand:

$$\begin{aligned} \frac{1}{k^2} &= \frac{1}{k_t^2 (1 + \frac{x_k}{x_l} \frac{(k-l)_t^2}{k_t^2})} \\ &\approx \frac{1}{k_t^2} \left( 1 + \frac{2(k_t l_t)}{k_t^2} \frac{x_k}{x_l} + \dots \right). \end{aligned} \quad (36)$$

Combining the correction term on the rhs with (35) we obtain the double logarithmic contribution:

$$k^2 (sl_t) \frac{2(k_t l_t)}{k_t^2} \text{Tr}[\gamma_5 x \hat{p} \gamma_\nu \hat{k} \gamma_\mu] = \frac{k^2 l_t^2}{2} \text{Tr}[\gamma_5 x_k \hat{p} \gamma_\nu \hat{s} \gamma_\mu], \quad (37)$$

which exactly cancels the contribution from (34). This shows that, in fact, also at the two loop level the structure  $(sl)\gamma_5 \hat{l}$  leads to odd signature only.

From this discussion the following general pattern emerges. As we have shown before, in a general ladder diagram of Fig.6 each rung reproduces the tensor structure  $\gamma_5 \hat{s}$  and creates, in addition, the beginning a new branch. Following the first branch, we have a mixture of even and odd signature, whereas the new branch belongs to odd signature only. As a general result, during the evolution along any of these branches signature is conserved. For the first branch, the final trace at the top of the ladder separates even and odd signature parts. In order to include the nonladder graphs in the correct way we have to discuss even and odd signature branches separately.

Let us first follow the odd signature part of the branch ( $\gamma_5 \hat{s}^\perp$ ): at each step it creates a new branch of pure odd signature with the spin structure  $\gamma_5 \hat{k}$ . For any odd signature branch in Fig.7

(say, with the structure  $\gamma_5 \hat{s}^\perp$  up to the  $j$ -th rung, and with the structure  $\gamma_5 \hat{k}$  above the  $j$ -th rung) we have the well-known infrared evolution equation which takes into account both the ladder graphs and the double logarithmic non-ladder bremsstrahlung contributions. For each such branch we have the same result as for  $g_1$ :

$$g_1(x, Q^2) = \frac{e_q^2}{2} \int_{-i\infty}^{+i\infty} \frac{d\omega}{2\pi i} \left(\frac{1}{x}\right)^\omega \left(\frac{Q^2}{\mu^2}\right)^{f^{(-)}(\omega)/2} \frac{\omega}{\omega - f^{(-)}(\omega)/2}. \quad (38)$$

Here  $f^{(-)}(\omega)$  denotes the quark scattering amplitude:

$$f^{(-)}(\omega) = \omega - \sqrt{\omega^2 - \frac{2C_F}{\pi} \alpha_s^L + \frac{2C_F}{2\pi^2 \omega} \alpha_s^N f_8^{(+)}}. \quad (39)$$

The non-ladder diagrams are contained in the function

$$f_8^{(+)} = 4\pi N \alpha_s^N \frac{\partial}{\partial \omega} \ln \left( e^{z^2/4} D_p(z) \right), \quad (40)$$

where

$$z = \omega / \sqrt{\frac{N \alpha_s^N}{2\pi}} \quad (41)$$

(for further details see the Appendix). Once we reach the top of the ladder, we ‘close’ with the trace. The odd signature part is induced by the  $\hat{Q}$  part.

The correct counting of these odd signature branches is done in the following way. New branches are created only at ladder rungs. Non-ladder gluons never change the spin structure. We therefore have to count the number of rungs, and we do this by formally distinguishing between  $\alpha_s^L$  (the strong coupling multiplying a gluon rung) and  $\alpha_s^N$  (the strong coupling multiplying a nonladder gluon). This procedure turns the  $\alpha_s$  in the second term in the square root expression (39) into  $\alpha_s^L$ , whereas the  $\alpha_s$  in the third term of this square root, as well as in (40) and (41) becomes  $\alpha_s^N$ . As in (27), the number of new odd signature branches is obtained by applying the derivative operator  $-\alpha_s^L \frac{\partial}{\partial \alpha_s^L}$  onto  $g_1$ . In addition we have a contribution  $g_1$  from the main  $\gamma_5 \hat{s}$  branch. So altogether we find:

$$g_{\perp}^{odd} = \left( 1 - \alpha_s^L \frac{\partial}{\partial \alpha_s^L} \right) g_1|_{\alpha_s^L = \alpha_s^N = \alpha_s}. \quad (42)$$

This generalizes the odd signature part of (28), in that it includes the nonladder diagrams.

Next the even signature branch: it is contained only in the tensor structure  $\gamma_5 \hat{s}^\perp$ , and it is projected out by the  $\hat{k}$  part of the final trace. Its coefficient is by a factor 2 smaller than in the usual  $g_1$  calculation (since for the longitudinal photon one can replace  $\hat{Q}$  by  $-2x\hat{p}$ ). For this branch nonladder graphs cancel, so our result equals:

$$-\frac{1}{2} g^L. \quad (43)$$

(cf.(25)).

Finally, in the one loop contribution we have to add also the nonladder graph. In section 3 we have argued that in the diagrams of Fig.3d the two cuts across  $q_1$  and  $q_2$  cancel against each other.

However, in the first loop (Fig.2b,c), where the cut of the quark  $q_1$  line corresponds to  $x = 1$  and where we have the term  $\hat{p}\gamma_5\hat{s}$  in the initial quark density matrix, the usual cancellation between the real and virtual DL contributions does not work. Hence within the DLA we have to take care about the non-ladder diagrams Fig.2b,c. They give a contribution of the form:

$$\frac{3e_q^2 C_F}{8\pi} \alpha_s. \quad (44)$$

Now we can add all contributions, (42), (43), and (44). Since this result for the transverse polarization is proportional to the sum  $g_1 + g_2$ , we have to subtract the known result for  $g_1$ . We obtain:

$$g_2(x, Q^2) = -\frac{1}{2}g^L(x, Q^2) - \left( \alpha_s^L \frac{\partial}{\partial \alpha_s^L} g_1 \right) \Big|_{\alpha_s^L = \alpha_s^N = \alpha_s} + \frac{3e_q^2 C_F}{8\pi} \alpha_s. \quad (45)$$

Eq.(45) is our final result for  $g_2$ . It differs from the one obtained in [27] in several respects: in [27] crossing properties and non-ladder graphs had not been analysed. As a result, the even signature contribution  $g^L$  was omitted, and the second term did not correctly contain the nonladder graphs. Finally, the last term representing the one loop contribution was missing.

## 6 The Wandzura-Wilczek Relation

Let us finally comment on the Wandzura-Wilczek (WW) relation (1) and see how our results fit into this equation. Since the WW-relation was proven to be valid only for the twist-2 part of  $g_2$  but not for the whole structure function  $g_2$ , we do not expect that our DLA result for the small- $x$  behavior satisfies this identity. It is instructive to illustrate the role of the twist-3 part in lowest order perturbation theory. For small  $\alpha_s \ll 1$  both functions  $g_1$  and  $g_2$  behave as  $\text{const} + O(\alpha_s \ln Q^2) + O(\alpha_s \ln \frac{1}{x})$ , but for  $g_2$  the "const" vanishes, and the term  $O(\alpha_s)$  goes to zero proportional to  $x$ . Therefore, at the order  $\alpha_s$ , the integral on the r.h.s. of eq.(1) contains one extra  $\ln 1/x$ , which on the lhs can be contained only in the twist-2 part of  $g_2$ . But since the full  $g_2$  (eq.(19)) has no such a logarithm at this order of the coupling  $\alpha_s$ , this logarithm in the twist-2 part must be cancelled by a similar logarithm in the twist-3 part of  $g_2$ . This illustrates that in the small- $x$  region the twist-2 and twist-3 contributions of  $g_2$  cannot be separated in an easy way.

The systematic construction of the twist-2 and twist-3 operators has, for example, been described in [16]. The  $n$ -moment of the spin dependent structure function is given by the expectation value of the operator:

$$R = \left( \frac{2i}{Q^2} \right)^n Q'_{\mu_1} \dots Q'_{\mu_n} [\bar{\psi} \gamma_5 \gamma_\sigma D_{\mu_1} \dots D_{\mu_n} \psi] , \quad (46)$$

where the matrix  $\gamma_\sigma$  is "directed" in accordance with the quark spin vector  $s_\sigma$ . In order to separate the twist-2 and twist-3 components, we have to symmetrize. The most symmetric operator

$$R_1 = \left( \frac{2i}{Q^2} \right)^n Q'_{\mu_1} \dots Q'_{\mu_n} \mathbf{S}_{\{\sigma, \mu_1, \dots, \mu_n\}} [\bar{\psi} \gamma_5 \gamma_\sigma D_{\mu_1} \dots D_{\mu_n} \psi] \quad (47)$$

gives the leading, twist-2 contribution, while the

$$R_2 = \left( \frac{2i}{Q^2} \right)^n Q'_{\mu_1} \dots Q'_{\mu_n} \mathbf{A}_{[\sigma, \mu_1]} \mathbf{S}_{\{\mu_1, \dots, \mu_n\}} [\bar{\psi} \gamma_5 \gamma_\sigma D_{\mu_1} \dots D_{\mu_n} \psi] \quad (48)$$

corresponds to twist-3 (in eqs. (47) and (48) the symbols  $\mathbf{S}_{\{\mu_1, \dots, \mu_n\}}$  and  $\mathbf{A}_{[\sigma, \mu_1]}$  denote symmetrization and antisymmetrization with respect to the indices  $\{\mu_1, \dots, \mu_n\}$  and  $[\sigma, \mu_1]$ , respectively.) In the case of longitudinal polarization of the target quark the vector  $s_\sigma$  has the same longitudinal direction as the derivative  $D_\mu$ . Therefore the operator  $R_2$  does not contribute to the longitudinal polarization, and twist-3 does not appear. This situation changes when one deals with the transverse spin vector  $s_\sigma^\perp$ . Now the twist-2 contribution is given by the operator  $R_1$ , e.g. for  $n = 1$ :

$$\langle h | R_1 | h \rangle \sim Q'_\mu \langle h | \bar{\psi} \gamma_5 \gamma_\sigma^\perp (x p_\mu) \psi | h \rangle + Q'_\mu \langle h | \bar{\psi} \gamma_5 \gamma_\mu k_\sigma^\perp \psi | h \rangle, \quad (49)$$

and  $R_2$  becomes nonzero:

$$\langle h | R_2 | h \rangle \sim Q'_\mu \langle h | \bar{\psi} \gamma_5 \gamma_\sigma^\perp (x p_\mu) \psi | h \rangle - Q'_\mu \langle h | \bar{\psi} \gamma_5 \gamma_\mu k_\sigma^\perp \psi | h \rangle. \quad (50)$$

Now it is easy to see that the Wandzura-Wilczek relation is valid only for the twist-2 part of the structure function. It follows just from the relativistic invariance [16]; both sides are the matrix elements of the different components of the same twist-2 tensor operator.

The results of our calculation show that at small  $x$  the WW-relation looks rather artificial. In particular, the twist-3 contribution is by no means small. For the first-loop we have shown that the twist-3 contribution adds an extra term to  $g_2$ , which is even more singular at  $x \rightarrow 0$ . For the case of the one loop  $O(\alpha_s)$  contribution the situation (and the behaviour of the moments of twist-2 and twist-3 operators) has been discussed in detail in [11]. In the context of the MIT-bag model, a large twist-3 contribution to  $g_2(x)$  has been emphasized in [28]. As a result, we conclude that the WW relation cannot be used to estimate the full function  $g_2$  at small  $x$ .

## 7 Conclusion

In this paper we have calculated the small  $x$  asymptotics of the spin dependent structure function  $g_2$  in the double log approximation of perturbative QCD. We started with the one and two loop approximations, and we then succeeded to perform the sum over all orders in  $\alpha_s$ . The asymptotic small  $x$  behaviour of  $g_2$  is governed by the rightmost singularity in the  $\omega$ -plane at  $\omega = \omega_0 \simeq \sqrt{2C_F\alpha_s}$  ( $\simeq 0.4$  at  $\alpha_s = 0.2$ ). This is the square root singularity of  $qq$ -amplitude  $f_0(\omega)$ . As a result, at  $x \rightarrow 0$  the structure function  $g_2$  behaves as

$$g_2 \propto x^{-\omega_0} \left( \frac{Q^2}{\mu^2} \right)^{\omega_0/2} \quad (51)$$

with the anomalous dimension  $\gamma = \omega_0/2$ .

There is an interesting difference between our transverse distributions  $g_\perp(x, Q^2)$  (or  $g_2(x, Q^2)$ ) and the structure function  $h_1(x, Q^2)$  which describes the evolution of the operator  $\langle \gamma_5 \hat{Q}' \hat{s}^\perp \rangle$ . The small  $x$  behaviour of  $h_1(x, Q^2)$  was studied in [29]. In the case of  $g_2(x, Q^2)$  the double logarithmic ladder evolution has been found to take place near the singularity at  $j = 0$ , and  $g_2(x, Q^2) \sim x^{-\lambda}$ , with  $\lambda = \omega_0 \sim O(\sqrt{\alpha_s}) > 0$ . In  $h_1$ , on the other hand, the situation is quite different. Namely due to the fact that after the interaction with the  $s$ -channel gluon the spin structure vanishes ( $d_{\alpha\beta}\gamma_\alpha\gamma_5\hat{Q}'\hat{s}^\perp\gamma_\beta = -4(\hat{Q}s^\perp)\gamma_5 = 0$ ), the leading  $\sim (1/x)^0$  part of the function  $h_1$  has no DL evolution at all, i.e. the singularity at  $j = 0$  in the complex angular momentum plane does not contribute. Only the singularity at  $j = -1$  leads to the DL contribution, and the small- $x$  behavior of  $h_1$  is suppressed by one power of  $x$  compared to  $g_2$ :  $h_1(x, Q^2) = x f(\alpha_s \ln^2 1/x, \alpha_s \ln 1/x \ln Q^2)$ .

Let us recall that in this paper, as a first step, we have calculated  $g_2$  only for deep inelastic scattering on a quark target, i.e. only the evolution of the  $g_2$  structure function within the perturbative QCD. Nothing has been said about the initial distribution of  $g_2$  coming from the confinement region. On the other hand, in ref.[30] it has been argued that, within the parton model, the intrinsic transverse momenta of the quarks (inside a proton) should lead to a non-zero value of  $g_2$  which satisfies the Wandzura-Wilczek relation (1). However, in order to reproduce the relation (1) one has to use very large transverse momenta ( $k_t^2 \sim Q^2$ ; as the integral over  $k_t^2$  corresponds in [30] to the integration over  $dy$  in eq.(1) with  $dy = dk_t^2/2m\nu$ ). Within the parton model such large values of intrinsic momenta  $k_t$  look somewhat unnatural.

## 8 Appendix

Here we will review the structure of the odd signature quark-quark amplitude and show where the QCD coupling ( $\alpha_s = g^2/4\pi$ ) corresponds to the ladder, and where it belongs to the non-ladder contributions. We have to separate these contribution in order to take the derivative with respect to the ladder coupling  $\alpha_s^L$  in the final expresion (45).

The well known equation for the odd signature quark-quark amplitude in the  $\omega$  representation has the form[13]

$$f_0^{(-)}(\omega) = \frac{N^2 - 1}{2N} \frac{g_L^2}{\omega} + \frac{1}{8\pi^2\omega} \left( f_0^{(-)}(\omega) \right)^2 - \frac{N^2 - 1}{N} \frac{g_{NL}^2}{4\pi^2} \frac{1}{\omega^2} f_8^{(+)}(\omega) \quad (1)$$

where the first two terms in the r.h.s. generate the ladder contribution while the last one corresponds to the non-ladder graphs.

This last term on the rhs is due to the signature changing contributions of gluon bremsstrahlung which lead us to define an even-signature quark quark scattering amplitude  $f_8^{(+)}(\omega)$  with color octet quantum number in the t-channel. This amplitude has the infrared evolution equation

$$f_8^{(+)}(\omega) = -\frac{g_L^2}{2N\omega} + \frac{1}{8\pi^2\omega} \left( f_8^{(+)}(\omega) \right)^2 + \frac{Ng_{NL}^2}{8\pi^2\omega} \frac{d}{d\omega} f_8^{(+)}(\omega) \quad (2)$$

Its solution follows from the discussion given in [13]: using the transformation

$$f_8^{(+)}(\omega) = Ng_{NL}^2 \frac{\partial}{\partial \omega} \ln u(\omega) \quad (3)$$

the Riccati equation (3) is equivalent to the linear differential equation

$$\frac{d^2 u}{dz^2} - z \frac{du}{dz} - \frac{1}{2N^2} \frac{g_L^2}{g_{NL}^2} u = 0 \quad (4)$$

where

$$z = \omega/\omega_0, \quad \omega_0 = \sqrt{Ng_{NL}^2/8\pi^2}. \quad (5)$$

After a trivial transformation this equation is solved by a parabolic cylinder function. As a result,  $f_8^{(+)}$  has the form:

$$f_8^{(+)}(\omega) = Ng_{NL}^2 \frac{d}{d\omega} \ln \left( e^{z^2/4} D_p(z) \right) = Ng_{NL}^2 \frac{d}{d\omega} \ln \left( \int_0^\infty dt t^{-1-p} e^{-zt - t^2/2} \right) \quad (6)$$

with

$$p = -\frac{1}{2N^2} \frac{g_L^2}{g_{NL}^2}. \quad (7)$$

With this we return to (1) and obtain for  $f_0^{(-)}$ :

$$f_0^{(-)} = 4\pi^2\omega \left( 1 - \sqrt{1 - \frac{(N^2 - 1)}{4\pi^2 N\omega^2} [g_L^2 - \frac{g_{NL}^2}{2\pi^2\omega} f_8^{(+)}(\omega)]} \right) \quad (8)$$

(the minus sign in front of the square root follows from the requirement that, for large  $\omega$ , the solution has to match the Born approximation).

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## References

- [1] G.Altarelli and G.Parisi, Nucl.Phys. **B126** (1977) 298.
- [2] M.A.Ahmed and E.G.Ross, Nucl.Phys. **B111** (1976) 441.
- [3] R.Mertig and W.L.Van Neerven, Z. Phys. **C70** (1996) 637.
- [4] J.Bartels,B.I.Ermolaev and M.G.Ryskin, DESY-95-124, Z. Phys. **C70** (1996) 273.
- [5] J.Bartels, B.I.Ermolaev and M.G.Ryskin, DESY-96-025, Z.Phys. **C7**
- [6] M.Anselmino, A.V.Efremov and E.Leader, Phys.Rep. **261** (1995) 1.
- [7] B.L.Ioffe V.A.Khoze and L.N.Lipatov, "Hard Processes", North-Holland, Amsterdam, 1984.
- [8] S.Wandzura and F.Wilczek, Phys.Lett. **B72** (1977)) 195.
- [9] A.J.G.Hey and J.E. Mandula, Phys.Rev. **D5** (1972) 2610.
- [10] G.Altarelli, B.Lampe, P.Nason, G.Ridolfi, Phys.Lett. **B334** (1994) 187.
- [11] A.Harindranath and Wei-Min Zhang, Phys.Lett. **B408** (1997) 347.
- [12] R.L.Heimann, Nucl.Phys. **B64** (1973) 429.
- [13] R.Kirschner and L.N.Lipatov, Nucl.Phys. **B213** (1983) 122.
- [14] J.Kodaira et al., Phys. Rev. **D20** (1979) 627; Nucl. Phys. **B159** (1979) 99.
- [15] E.V.Shuryak and A.I.Vainshtein, Nucl. Phys. **B199** (1982) 951; Nucl. Phys. **B201** (1982) 141

- [16] A.P.Bukhvostov, E.A.Kuraev and L.N.Lipatov, Sov.Phys.JETP **60** (1984) 22.
- [17] P.G.Ratcliffe, Nucl. Phys. **B264** (1986) 493.
- [18] I.I.Balitsky and V.M.Braun, Nucl. Phys. **B311** (1989) 541.
- [19] X.Ji and C.Chou, Phys. Rev. **D42** (1990) 3637.
- [20] J.Kodaira, Y.Yasui and T.Uematsu, Phys.Lett. **B344** (1995) 348; J.Kodaira, Y.Yasui, K.Tanaka and T.Uematsu, Phys.Lett. **B387** (1996) 855.
- [21] A.Ali, V.M.Braun and G.Hiller, Phys. Lett. **B266** (1991) 117.
- [22] V.N.Gribov, V.G.Gorshkov, G.V.Frolov and L.N.Lipatov, Sov.J.Nucl.Phys. **6** (1967) 95.
- [23] B.I.Ermolaev, S.I.Manayenkov and M.G.Ryskin, Z.Phys. **C69** (1996) 259.
- [24] V.V. Sudakov, Sov.Phys.JETP **3** (1956) 65.
- [25] D.Amati, S.Fubini, A.Stangellini, Nuovo Cim. **26** (1962) 826.
- [26] S.Mandelstam, Nuovo Cim. **30** (1963) 1113, 1127.
- [27] B.I.Ermolaev and S.I.Troyan, hep-ph/9703384.
- [28] R.L.Jaffe and X.Ji, Phys. Rev. **D43** (1991) 724.
- [29] R.Kirschner et al., Z.Phys. **C74** (1997) 501.
- [30] J.D.Jackson, G.G.Ross and R.G.Roberts, Phys.Lett. **B226** (1989) 159;  
R.G.Roberts and G.G.Ross, Phys. Lett. **B373** (1996) 235.